

A Lax pair of a lattice equation whose entropy vanishes

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Abstract

In this paper, we present multi parametric quadgraph equations which are consistent around the cube. These equations are obtained by applying a ‘double twist’ to known integrable equations. Furthermore, we perform a limit to one of these equations to derive the non-symmetric equation which is Equation 19 in [7]. As a result, we obtain a novel Lax pair of this equation.

1 Introduction

There are various definitions of integrability for both continuous and discrete integrable systems. One of the common definitions is the notion of having a Lax representation. Integrable non-linear equations arise as the compatibility condition of two linear systems called a Lax pair. The origins lie in the inverse scattering method introduced by Gardner, Greene, Kruskal and Miura [6] to solve the initial value problem for Korteweg-de Vries (KdV) equation. In this paper, we consider only lattice integrable equations on a square

$$Q(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = 0, \quad (1)$$

whose Lax pair is often given by a pair of matrices. In this equation, the field variable, u , depends on lattice variables l and m where $(l, m) \in \mathbb{Z}^2$.

The definition of integrability associated with having a Lax pair is quite simple. Nevertheless, the question arising is how to construct a Lax pair of a given equation. There is a systematic way of constructing a Lax pair of equations on quad graphs that satisfy consistency around the cube (CAC) and other properties [1, 9, 10]. The CAC property gives us a Lax pair for the equations directly. There is a list of Lax pairs for many CAC equations and systems, including equations in the Adler-Bobenko-Suris (ABS) list [1, 2] given in [3]. However, in general if a lattice equation is not CAC, it would be difficult to find its Lax pair if it exists.

Integrable systems are often associated with ‘low complexity’. One of the means to measure the complexity of a discrete system is called algebraic entropy [5, 13, 14]. It is believed that discrete integrable mappings and lattice equations have vanishing entropy. In other words, given a map we can use initial values to iterate the map. For an integrable map, one would get a polynomial growth of degrees of the map. It also holds for integrable lattice equations. Suppose that we can solve uniquely at any vertex and obtain a rule to propagate new vertices. Vanishing entropy can be seen through factorisation and large cancellations. As a consequence, it gives a slow growth of degrees of iterations. Vanishing entropy for lattice equations has been used as a simple integrability test

for a decade. However, it has only been proven recently for certain lattice equations subject to a conjecture [12].

On the other hand, ‘low complexity’ has been used in [7] to search for integrable lattice equations on a square. The authors searched for multi-affine equations on a quad-graph with a certain factorisation property. Two of the highlights in their paper are Equation 19 and Equation 25. These equation are given as follows

$$\text{E19 : } (x - v)(u - y) + r_4(x - y) + r_2(u - v) + s = 0, \quad (2)$$

and

$$\text{E25 : } xy + uv + p_3(xu + vy) - (p_3 + 1)(xv + uy) + r_4(y - x) + r_2(u - v) - (s(p_3 + 1) + r_4)(p_3s + r_4) + sr_2 = 0, \quad (3)$$

where we have used the short notation given in [1], i.e. we denote $u_{l,m} = x, u_{l+1,m} = u, u_{l,m+1} = v$ and $u_{l+1,m+1} = y$. These two equations have vanishing entropy; therefore, they are likely to be integrable. Equation E25 given by (3) is related to Q_1 via a non-autonomous transformation except for $p_3 = 0$ or $p_3 = -1$. However, equation E19 (see equation (2)) seems to be new as it is not symmetric. This makes it hard to find its Lax pair (if it exists) as we can not put this equation on a cube. So far, not much has been known about this equation. Furthermore, at the first glance, we can see that equation E19 is a special case of equation E25. In fact, it falls into an exceptional case of the latter one where $p_3 = -1$. Thus, it seems that it would be hard to connect Q_1 and equation E19 directly but one still hopes to establish connections through a limit procedure.

Recently, there have been interesting results obtained by Ormerod et al in [11]. The authors introduced the notion of a twist matrix which is associated with a twisted reduction T where T is taken as a point symmetry of quad-graph equations. If we could find a twist matrix corresponding to T , we will obtain Lax pairs of twisted reductions which are generalisation of periodic reductions. In this paper, we use this idea at the level of lattice equations. In fact, the twist acts only on two vertices on one edge of an equation on a square. It gives different transformations on four points of a square. In general, those transformations do not preserve nice properties of integrable equations such as Lax pairs and vanishing entropy.

We note that a non-autonomous transformation sometimes can be written as a twist. Therefore, it suggests that we might be able to connect Q_1 with equation E25 via some twists. In this paper, we establish a connection between these two equations. This will help to find autonomous Lax pairs for equations E25 and E19.

This paper is organised as follows. In section 2, we will set up some definitions that will be needed for the rest of the paper. In the next section, we will present some CAC equations (including equation E25) with multi-parameters. These equations are obtained through a ‘double twist’. We note that the CAC property provides us with a Lax pair. However, the double twist also gives us a Lax pair for the corresponding equation. In particular, we present a Lax pair of equation E25 with a different parametrisation. It has been done through two twists of Q_1 . In section 4, we present in detail how to derive a Lax pair for equation E19 from the Lax pair of equation E25.

2 The setting

In this section, we introduce some notions that will be used in this paper. We consider equation (1) on a quad graph. The equation is called integrable if it arises as the compatibility condition of the following system of linear equations

$$\begin{aligned} \phi_{l+1,m} &= L_{l,m}\phi_{l,m}, \\ \phi_{l,m+1} &= M_{l,m}\phi_{l,m}, \end{aligned} \quad (4)$$

where ϕ is a vector, L and M are matrices that depend on spectral parameter, λ . Generally L depends on $u_{l,m}, u_{l+1,m}$ and M depends on $u_{l,m}, u_{l,m+1}$. Thus, equation (1) is integrable if it satisfies the compatibility condition

$$L_{l,m+1}M_{l,m} - M_{l+1,m}L_{l,m} = 0. \quad (5)$$

Definition 1. A pair of matrices, L and M , satisfying (5) if equation (1) satisfied is called a Lax pair of equation (1).

For example, equation Q_1 which is given by cf.[1]

$$p(x-v)(u-y) - q(x-u)(v-y) + \delta^2 pq(p-q) = 0 \quad (6)$$

has the following Lax pair cf. [3]

$$L(x, u, p) := \begin{pmatrix} \frac{-ku+kx+pu}{\delta p-u+x} & \frac{\delta^2 pk^2 - \delta^2 p^2 k - pxu}{\delta p-u+x} \\ \frac{p}{\delta p-u+x} & \frac{-ku+kx-px}{\delta p-u+x} \end{pmatrix}, \quad M(x, v, q) := \begin{pmatrix} \frac{-kv+kx+qv}{\delta q-v+x} & \frac{\delta^2 qk^2 - \delta^2 q^2 k - qxv}{\delta q-v+x} \\ \frac{q}{\delta q-v+x} & \frac{-kv+kx-qx}{\delta q-v+x} \end{pmatrix}. \quad (7)$$

We perform a transformation, $\phi_{l,m} \rightarrow G_{l,m}\phi_{l,m}$, where G is any non-singular matrix. This gives $\tilde{L}_{l,m} = G_{l+1,m}L_{l,m}G_{l,m}^{-1}$ and $\tilde{M}_{l,m} = G_{l,m+1}M_{l,m}G_{l,m}^{-1}$, and the equation (1) is invariant under this transformation. We call this transformation a gauge transformation and G is a gauge matrix. In fact, it just gives us a different form of a Lax representation for equation (1).

Recall that a transformation T is called a point symmetry of (1) if

$$Q(T(u_{l,m}), T(u_{l+1,m}), T(u_{l,m+1}), T(u_{l+1,m+1})) = 0. \quad (8)$$

In [11], the authors introduced the quasi-periodicity $u_{l+s_1, m+s_2} = T(u_{l,m})$ as the (s_1, s_2) twisted reduction. A direct method to obtain a Lax pair of these reductions was presented. Given an integrable equation Q with a Lax pair (L, M) , we consider the following equation

$$Q(T(u_{l,m}), T(u_{l+1,m}), u_{l+1,m}, u_{l+1,m+1}; p, q) = 0, \quad (9)$$

where T is a point transformation such that the above equation has a vanishing entropy. It is noted that T does not need to be a point symmetry of Q . We have

$$M(T(u_{l+1,m})u_{l+1,m+1}, q) L(T(u_{l,m}), T(u_{l+1,m}, p)) = L(u_{l,m+1}, u_{l+1,m+1}, p) M(T(u_{l,m}), u_{l,m+1}, q).$$

We look for a twist matrix S that satisfies

$$L(T(u_{l,m}), T(u_{l+1,m}, p)) S(u_{l,m}, p) = S(u_{l+1,m}, p) L(u_{l,m}, u_{l+1,m}, p).$$

Therefore, we have

$$M(T(u_{l+1,m})u_{l+1,m+1}, q) S(u_{l+1,m}, p) L(u_{l,m}, u_{l+1,m}, p) S^{-1}(u_{l,m}, q) = L(u_{l,m+1}, u_{l+1,m+1}, p) M(T(u_{l,m}), u_{l,m+1}, q).$$

This implies that $(L(u_{l,m}, u_{l+1,m}, p), M(T(u_{l,m}), u_{l,m+1}, q) S(u_{l,m}, p))$ is a Lax pair for (9). We note that for many cases, twist matrices might not exist.

3 CAC equations with multi-parameters

In this section, we present some equations which are CAC. These equations are obtained by performing a double twist on Q_1 and H_1 . Using a twist matrix we obtain Lax pairs for these equations. We use Q_1 as an example. We present a Lax pair for equation E25 (equation (3)) through a connection with Q_1 . As mentioned above, when $p_3 \neq 0, -1$, one can bring this equation to Q_1 via a transformation that depends on lattice variables l and m . Therefore, we can obtain a non-autonomous Lax pair of this equation from a Lax pair of Q_1 . However, our aim is to find a Lax pair that does not depend on l and m explicitly.

We first rewrite equation E25 with a different parametrisation as follows

$$p(x - v + c_1)(u - y + c_1) - q(x - u + c_2)(v - y + c_2) + \delta^2 pq(p - q) = 0. \quad (10)$$

This equation is a ‘double twist’ of Q_1 in the sense that it is given by

$$Q(T_2(T_1(x)), T_1(u), T_2(v), y, p, q) = 0, \quad (11)$$

where $T_1(x) = x + c_1$ and $T_2(x) = x + c_2$.

For a single twist T_1 , we first look for a constant twist matrix S_1 . By solving the equation $L(T_1(x), T_1(u), p)S = SL(x, u, p)$, where L is given by (7), one gets

$$S_1 = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}. \quad (12)$$

Therefore, we obtain a new \mathcal{M} matrix of the equation $Q(T_1(x), T_1(u), v, y, p, q) = 0$, where

$$\mathcal{M} = \begin{pmatrix} \frac{qv - kv + kx + kc_1}{\delta q - v + x + c_1} & \frac{-\delta^2 q^2 k + \delta^2 q k^2 - qvx - kv c_1 + kx c_1 + k c_1^2}{\delta q - v + x + c_1} \\ \frac{q}{\delta q - v + x + c_1} & \frac{-qx - kv + kx + kc_1}{\delta q - v + x + c_1} \end{pmatrix}. \quad (13)$$

One can then apply the second twist T_2 to find a new Lax matrix \mathcal{L} . However, we note that the twists T_1 and T_2 are commutative, i.e. $T_1(T_2(x)) = T_2(T_1(x))$. Therefore, we can also apply the twist T_2 first. This gives us a new matrix \mathcal{L} for $Q(T_2(x), T_2(u), v, y, p, q) = 0$, where

$$\mathcal{L} = \begin{pmatrix} \frac{pu - ku + kx + kc_2}{\delta p - u + x + c_2} & \frac{-\delta^2 p^2 k + \delta^2 p k^2 - pxu - kuc_2 + kx c_2 + k c_2^2}{\delta p - u + x + c_2} \\ \frac{p}{\delta p - u + x + c_2} & \frac{-px - ku + kx + kc_2}{\delta p - u + x + c_2} \end{pmatrix}. \quad (14)$$

It is easy to check that $(\mathcal{L}, \mathcal{M})$ is a Lax pair for equation (10). We note that equation (10) is consistent around the cube with multi-parameters $((p, c_2), (q, c_1))$. Thus, we can obtain a Lax pair directly from CAC. It actually gives us a Lax pair with two spectral parameters. By taking one of these two spectral parameters to be 0, we obtain the Lax pair $(\mathcal{L}, \mathcal{M})$ given by (14) and (13).

Remark 2. We give here two more CAC equations.

- Similarly, when $\delta = 0$, one can use two twists $T_3(x) = \alpha_1 x$ and $T_4(x) = \beta_1 x$ on Q_1 . A twist matrix is given by

$$S_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}. \quad (15)$$

This gives us a Lax pair of a multi-parametric cross-ratio equation cf. [8]

$$p\beta_1(\alpha_1 x - v)(\alpha_1 u - y) = q\alpha_1(\beta_1 x - u)(\beta_1 v - y). \quad (16)$$

This equation satisfies the 3D consistency and tetrahedron properties [1] and has two parameters in each direction. Moreover, it is actually a special case of Equation 16 in [7].

- Applying the twist T_1 and T_2 to H_1 , we obtain the twist matrix

$$S_c = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \quad (17)$$

and the following equation

$$(x - y)(u - v) + (c_1 + c_2)(u - v) + (c_1 - c_2)(x - y) + c_1^2 - c_2^2 - p + q = 0. \quad (18)$$

This equation is a slightly more general form of Equation B_1 given in [4]. We have checked that this equation is consistent around the cube with lattice parameters $((p, c_2), (q, c_1))$.

Remark 3. We note that all the equations which we obtain in this sections can be transformed to Q_1 and H_1 (respectively) via non-autonomous transformations.

4 Lax pair of Equation 19

In this section, we derive a Lax pair of Equation 19 given by (2) from the previous Lax pair of equation (10).

As discussed in the introduction, equation E19 is a special case of equation E25 with $p_3 = -1$. In the new form (10) of equation E25 if we take $q = 0$, we can get rid of the term $(x - u)(v - y)$ but the equation (10) will become trivial. Hence, one needs to avoid this problem by considering equation (2) as a limit of equation (10).

The process of deriving a Lax pair of this equation is given as follows.

- By dividing both sides of the equation (10) by $pq(p - q)$, we obtain the following equation

$$\frac{(x - v + c_1)(u - y + c_1)}{q(p - q)} - \frac{(x - u + c_2)(v - y + c_2)}{p(p - q)} + \delta^2 = 0. \quad (19)$$

- In order to get the term $(x - v)(u - y)$ and cancel the term $(x - u)(v - y)$, we choose p, q such that $q(p - q) \rightarrow 1$ and $p(p - q) \rightarrow \infty$. Thus, one can choose $q = \epsilon$ and $p = \epsilon + 1/\epsilon$. To obtain the linear and constant terms in the equation (2), we need to choose c_2 such that $\frac{c_2}{p(p - q)}$ and $\delta^2 - \frac{c_2^2}{p(p - q)}$ approach to constants as $\epsilon \rightarrow 0$. Therefore, we choose $c_2 = \beta/\epsilon^2$ and $\delta = \beta\sqrt{s\epsilon^2 + 1}/\epsilon$.
- Substituting these values into \mathcal{L} and \mathcal{M} given by (14) and (13), one would hope to obtain a Lax pair for equation E19 by taking the limit of these two matrices. However, there is a factor ϵ in a denominator of \mathcal{L}_{12} . To avoid this, we choose a spectral parameter to be ϵk . We then divide the matrices \mathcal{L}, \mathcal{M} by ϵ to cancel the factor ϵ and take the limit of these results as $\epsilon \rightarrow 0$. We obtain the following matrices

$$L_1 = \begin{pmatrix} \frac{\beta k + u}{2\beta} & \frac{\beta^2 k^2 - \beta^2 k s - 2\beta^2 k - \beta k u + \beta k x - u x}{2\beta} \\ \frac{1}{2\beta} & \frac{\beta k - x}{2\beta} \end{pmatrix} \quad (20)$$

and

$$M_1 = \begin{pmatrix} -\frac{k v - k x - k c_1 - v}{\beta + c_1 - v + x} & \frac{\beta^2 k^2 - \beta^2 k - k v c_1 + k x c_1 + k c_1^2 - v x}{\beta + c_1 - v + x} \\ \frac{1}{\beta + c_1 - v + x} & -\frac{k v - k x - k c_1 + x}{\beta + c_1 - v + x} \end{pmatrix} \quad (21)$$

These matrices are a Lax pair of the following equation

$$(x - v)(u - y) + (c_1 - \beta)(x - y) + (c_1 + \beta)(u - v) + c_1^2 + \beta^2 + \beta^2 s = 0, \quad (22)$$

which is Equation 19 given by Hietarinta and Viallet. By multiplying the matrix L_1 with 2β , taking $c_1 = \alpha$ and replacing $\beta^2 s$ with s , we obtain the following equation

$$(x - v)(u - y) + (\alpha - \beta)(x - y) + (\alpha + \beta)(u - v) + \alpha^2 + \beta^2 + s = 0, \quad (23)$$

which has the following Lax pair

$$\mathcal{L}_1 = \begin{pmatrix} \beta k + u & \beta^2 k^2 - 2\beta^2 k - \beta ku + \beta kx - ks - ux \\ 1 & \beta k - x \end{pmatrix}, \quad (24)$$

and \mathcal{M}_1 is obtained by replacing c_1 with α .

Using the gauge matrix

$$G = \begin{pmatrix} 1 & -\beta k - x \\ 0 & 1 \end{pmatrix}, \quad (25)$$

we get a simpler looking Lax pair

$$\tilde{\mathcal{L}}_1/k = \begin{pmatrix} 0 & -2\beta u - 2\beta^2 - s + 2\beta x \\ \frac{1}{k} & 2\beta \end{pmatrix}, \quad \tilde{\mathcal{M}}_1/k = \begin{pmatrix} \frac{\alpha - v + x - \beta}{\beta + \alpha - v + x} & \alpha - v + x - \beta \\ \frac{1}{k(\beta + \alpha - v + x)} & 1 \end{pmatrix}. \quad (26)$$

This Lax pair is useful for calculating integrals for reduced maps obtained as reductions of equation E19.

The Lax pair (\tilde{L}, \tilde{M}) gives us a conservation law

$$(F, G) = (\ln(s + 2\beta^2 + 2\beta(u - x)), \ln(\alpha - \beta - v + x) - \ln(\alpha + \beta - v + x)). \quad (27)$$

Remark 4. When $\alpha = \beta$ and $s = -2\beta^2$, equation E19 is a special case of the following equation whose entropy vanishes

$$\text{E18: } (p_3 v + x)(p_3 y + u) + r_3(p_3 v + u) = 0, \quad (28)$$

with $p_3 = -1$. This equation is Equation 18 given in [7]. we get a Lax pair for equation (28) with $p_3 = -1$ and $r_3 = 2\beta$ as follows

$$L = \begin{pmatrix} 0 & -2\beta u + 2\beta x \\ \frac{1}{k} & 2\beta \end{pmatrix}, \quad M = \begin{pmatrix} -\frac{(-x+v)}{2\beta - v + x} & -(-x + v) \\ \frac{1}{k(2\beta - v + x)} & 1 \end{pmatrix}. \quad (29)$$

5 Conclusion

In conclusion, this paper has presented for the first time a novel Lax pair for equation E19 given in [7]. It reinforces that equation E19 is actually integrable in the sense of having a Lax representation. This Lax pair was derived by using a double twist of Q_1 and taking a limit of this twisted equation. A double twist has also been used to derive multi-parametric versions of the cross ratio equation, H_1 equation and their Lax pairs. We would also like find a Lax pair for equation E18 given by (28). We have tried to find its Lax pair from a twist $T(x) = \lambda x$ and a Lax pair (29), but the twist matrix does not exist. Therefore, it leaves an open question whether equation E18 has a Lax pair. If it does, then how can one find it?

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